$\Rightarrow |\xi_j^{(n)} - \xi_j| < \epsilon/3 \quad \forall\, n \geq N \;\&\; \forall\, j \in \mathbb{N}.$ —— ①

In particular,

$|\xi_j^{(N)} - \xi_j| < \epsilon/3 \quad \forall\, j \in \mathbb{N}$ —— ②

Now $x_N = \langle \xi_j^N \rangle \in C \subset l^\infty$

$\Rightarrow \langle \xi_j^N \rangle$ is a cgt sequence.

$\Rightarrow \langle \xi_j^N \rangle$ is Cauchy

$\Rightarrow$ for the above $\epsilon > 0$, $\exists$ a +ve integer K_0 s.t.

$|\xi_j^N - \xi_k^N| < \epsilon/3 \quad \forall\, j, k \geq K_0$ —— ③

Consider $|\xi_j - \xi_k| = |\xi_j - \xi_j^N + \xi_j^N - \xi_k^N + \xi_k^N - \xi_k|$

$\leq |\xi_j - \xi_j^N| + |\xi_j^N - \xi_k^N| + |\xi_k^N - \xi_k|$

$< \epsilon/3 + \epsilon/3 + \epsilon/3$ (by ①, ② & ③)

$= \epsilon \quad \forall\, j, k \geq K_0$

$\Rightarrow |\xi_j - \xi_k| < \epsilon \quad \forall\, j, k \geq K_0$

$\Rightarrow \langle \xi_j \rangle$ is a Cauchy sequence in $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$)

$\Rightarrow \langle \xi_j \rangle$ is cgt ($\because \mathbb{K}$ is complete)

$\Rightarrow x = \langle \xi_j \rangle \in C$

$\Rightarrow x \in C$

$\Rightarrow \overline{C} \subset C$ (but $C \subset \overline{C}$ always)

$\Rightarrow C = \overline{C}$

$\Rightarrow C$ is closed in l^∞

$\Rightarrow C$ is complete

$\Rightarrow (C, \|\cdot\|_\infty)$ is a Banach space. *

(5) The sequence space c_0 is complete.

$\because c_0 = \{ x = \langle \xi_n \rangle : \langle \xi_n \rangle \longrightarrow 0 \}$

We know: $c_0 \subset l^\infty$ & l^∞ is complete.

We'll show that c_0 is closed in l^∞.

Let $x = \langle \xi_j \rangle \in \overline{c_0}$ be arb.

Since $x \in \overline{c_0}$

$\Rightarrow \exists$ a sequence $\langle x_n \rangle^{(*)}$ in c_0 s.t. $x_n \longrightarrow x$

Where $x_n = \langle \xi_j^n \rangle$